Logical Necessity and Logical Atomism: Lecture Ten
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1 Introduction

The problem with modal logic, according to its critics, is quantification into modal contexts, which is traditionally known as *de re* modality. By allowing such quantification, these critics argue, we become committed to essentialism, and perhaps a bloated universe of logically possible objects as well. The essentialism is avoidable, it is then claimed, but only by turning to a Platonic realm of individual concepts whose existence is no less dubious or problematic than logically possible objects. Moreover, basing one's semantics on individual concepts would in effect render all identity statements containing only proper names either necessarily true or necessarily false—i.e., there would then be no contingent identity statements containing only proper names.[1]

These claims are not true independently of what formal ontology we adopt. We need not be committed to any kind of essentialism in the modal logics based on tense logic, for example, nor need there be any commitment in that framework to logically possible objects or individual concepts. Indeed, as already noted in our second lecture, there need not even be a commitment to realia other than the objects that presently exist, or to objects that either did or do exist.

The claim that identity statements containing only proper names would be either necessarily true or necessarily false does not depend, moreover, on a commitment to individual concepts, but on whether or not proper names are "rigid designators," i.e., on whether a proper name is assumed to denote the same object in every possible world, or at any time, at which that object exists.

The commitment to essentialism that these critics have in mind, moreover, is not Aristotelian essentialism, even though Quine, who has been the most vocal

[1] Cf. Quine 1943, 1947, 1953.

of the critics, explicitly condemns it as such.[2] The examples Quine gives of essentialism, such as,

> the number nine is necessarily greater than seven, but
> nine is only contingently the number of planets, and
> everything is necessarily self-identical,

are not based on an ontology of natural kinds and have nothing to do with Aristotelian essentialism. Apparently, what Quine and the other critics of modal logic have in mind is either a logical essentialism based on logical modalities or a metaphysical essentialism based on some form of realism other than natural realism and some sort of logical modalities broadly understood.

Now there is an ontology in which all of these objections to modal logic, including especially those directed against *de re* modalities, can be shown to be completely false. This is the ontology of logical atomism, where there are no individual concepts and no possibilia other than the simple objects that exist in the actual world. In addition, it is probably only in logical atomism that logical necessity and possibility find their clearest, and perhaps their only, adequate explication. And yet, not only is logical essentialism false in logical atomism, but it is refutable as well. In other words, with respect to the logical modalities, a modal thesis of anti-essentialism is valid in logical atomism, and one consequence of this is that all *de re* logical modalities are reducible to *de dicto* logical modalities, and hence that there is no problem of *de re* logical modalities.

> Logical atomism:
>
> (1) There are no individual concepts and no possibilia.
>
> (2) Logical essentialism is refuted because the modal thesis of anti-essentialism is logically true in this framework.
>
> (3) All *de re* logical modalities are reducible to *de dicto* logical modalities.

These results do not mean that logical atomism provides the kind of formal ontology we should adopt, and in fact there are good reasons why just the opposite is the case. Nevertheless, logical atomism does provide the paradigm framework by which to understand logical necessity and possibility, and it shows why a logical essentialism based on this kind of necessity—as opposed, e.g., to a natural necessity—not only does not follow but is actually refuted.

Now opposed to logical atomism, but on a par with it in its referential interpretation of quantifiers and proper names, is Saul Kripke's semantics for what he calls metaphysical necessity.[3] There are no individual concepts in Kripke's semantics, and yet proper names are "rigid designators," which means that there can be no contingent identity statements containing only proper names. But

[2] Cf. Quine 1953, p. 173–174.
[3] Cf. Kripke, 1971, p. 150.

unlike what is meant by the clear and primary meaning of the phrase "all possible worlds" in logical atomism, and with respect to which logical necessity is interpreted, Kripke's semantics allows for a "cut-down" on the totality of possible worlds, so that the notion of "all possible worlds," and hence necessity, has a secondary meaning in his semantics.[4] Such a secondary meaning or "cut-down" on the notion of "all possible worlds," and therefore on necessity, amounts to an initial, but incomplete, step toward something like Aristotelian essentialism. The problem with this initial step, and why it is incomplete, is its failure to provide any ontological content—as opposed to a merely formal, set-theoretic structure—to what is meant by necessity and possibility. This problem of the secondary meaning of necessity, or "cut-down," on the notion of "all possible worlds," is only compounded, moreover, by adding to this semantics a relation of accessibility between possible worlds. What must then be explained, in other words, is not just the philosophical significance of the "cut-down" on the notion of "all possible worlds," but also the ontological content of the accessibility relation between possible worlds.

> The real problem of quantified modal logic for an ontology other than logical atomism, in other words, is to give an ontological account of the "cut-down" on the notion of "all possible worlds" and of the accessibility relation between possible worlds. The question is: can this really be done other than in tense logic or Aristotelian essentialism, both of which are contained in conceptual realism?

2 The Ontology of Logical Atomism

Reality, according to logical atomism, consists of the existence and nonexistence of atomic states of affairs, where the existence of a state of affairs is "a positive fact" and its nonexistence "a negative fact".[5] The actual world, in other words, consists of all that is the case, namely the totality of facts, whether positive or negative.[6] Every other possible world consists of the same atomic states of affairs that make up reality, except that what are positive facts in one world can be negative facts in another, with every possible combination of atomic states of affairs being realized in some possible world or other.[7] The totality of possible worlds, in other words, is completely determined by all the combinations of

[4] A secondary semantics for necessity stands to the primary semantics in essentially the same way that nonstandard models for second-order logic stand to standard models. See Cocchiarella 1975.

[5] Wittgenstein 1961, 2.06. For a fuller discussion of logical atomism as a formal ontology, see chapters 6 and 7 of Cocchiarella 1987.

[6] Wittgenstein 1961, 1. It is an issue as to whether the *Tractatus* allowed for negative facts. In any case, there are negative facts in Russell's version of logical atomism.

[7] If we did not include negative facts, then a world that contains none of the states of affairs that "exist" in the actual world—i.e., that would contain as positive facts all of the negative facts of the actual world—would be an empty world, a world devoid of all facts, and hence of all objects as well, and therefore not a possible world at all.

the existence or nonexistence of the states of affairs that make up reality. The direction of this determination is important. Atomic states of affairs do not have *being* (the-case-or-not-the-case) because they exist (are the case) in some possible worlds; rather, possible worlds are possible because they are resolvable into the atomic states of affairs that make up reality.

> The determination of the totality of possible worlds in terms of the atomic states of affairs that make up reality is what makes logical atomism a paradigm framework for the semantics of logical necessity.

Every atomic state of affairs is a configuration of objects, and therefore because every state of affairs is a positive or negative fact in each possible world, each possible world consists of the same totality of objects as every other possible world. There is no distinction, accordingly, between the existence and being of objects. In the ontological grammar of logical atomism, in other words, there is no distinction between the possibilist quantifiers $\forall$ and $\exists$ and the actualist quantifiers $\forall^e$ and $\exists^e$, and for that reason we will not include the latter in the formal ontology of logical atomism.

Ordinary proper names of natural language are not "logically proper names" in the framework of logical atomism. The things they name, if they name anything at all, are not the simple objects that are the constituents of atomic states of affairs. The names of ordinary language have a sense (*Sinn*), moreover, in so far as they are introduced into discourse with identity criteria, usually provided by a sortal common noun with which they are associated.[8] The logically proper names of logical atomism have no sense other than what they denote. In other words, in logical atomism, "a name means (*bedeutet*) an object. The object is its meaning (*Bedeutung*)."[9] Different identity criteria have no bearing on the simple objects of logical atomism, and (pseudo) identity propositions, strictly speaking, have no sense (*Sinn*), i.e., they do not represent an atomic state of affairs.

Semantically, what this comes to is that logically proper names, or objectual constants, are rigid designators; that is, their introduction into formal languages requires that the formula

$$(\exists x)\Box(a = x)$$

be logically true in the primary semantics for each objectual constant a.

Kripke also claims that proper names are rigid designators, but his proper names are those of ordinary language, and his necessity is metaphysical and not logical necessity. Nevertheless, in agreement with logical atomism the function of a proper name, according to Kripke, is simply to refer, and not to describe the object named[10]; and this applies even when we fix the reference of a proper

[8] Cf. Geach, 1980, p. 63f. Individual constants cannot be vacuous in logical atomism, moreover, which means that the free logic of the quantifiers $\forall^e$ and $\exists^e$ is to be excluded, as opposed to the logic of the possibilist quantifiers $\forall$ and $\exists$, which, as already noted is standard predicate logic.

[9] Wittgenstein 1961, 3.203.

[10] Kripke, 1971, p. 140.

name by means of a definite description—for the relation between a proper name and a description used to fix the reference of the name is not that of synonymy.[11]

Because logically proper names are rigid designators, there are no contingent identity statements involving only proper names. That is, where a, b are objectual constants,

$$(a = b) \to \Box(a = b)$$

$$(a \neq b) \to \Box(a \neq b)$$

are both understood to be logically true in logical atomism, and in Kripke's framework of metaphysical necessity as well. The fact that there can be no contingent identities or non-identities in logical atomism is reflected, moreover, in the logical truth of both of the formulas

$$(\forall x)(\forall y)[(x = y) \to \Box(x = y)],$$

$$(\forall x)(\forall y)[(x \neq y) \to \Box(x \neq y)]$$

in the semantics of logical atomism.[12] But then even in the framework of Kripke's metaphysical necessity (where quantifiers also refer directly to objects), an object cannot but be the object that it is, nor can one object be identical with another—a metaphysical fact which is reflected in the above formulas being valid in that framework as well.

Another observation about the ontology of logical atomism is that the number of objects in the world is part of its logical scaffolding.[13] That is, for each positive integer n, it is either logically necessary or impossible that there are exactly n objects in the world; and if the number of objects is infinite, then, for each positive integer n, it is logically necessary that there are at least n objects in the world.[14] This is true in logical atomism because every possible world consists of the same totality of objects.

One important consequence of the fact that every possible world (of a given logical space) consists of the same totality of objects is the logical truth of the Carnap-Barcan formula (and its converse)

$$(\forall x)\Box\varphi \leftrightarrow \Box(\forall x)\varphi.$$

Carnap, it should be noted, was the first to actually give a semantic argument justifying the logical truth of this principle.[15] The idea, in effect, is that any universally quantified sentence $(\forall x)\varphi$, no matter whether φ contains occurrences of modal operators or not, "is to be interpreted as a joint assertion for all values of the variable."[16]

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[11] Ibid., pp. 156f.

[12] See Carnap 1946 for the first clear recognition of the validity of these noncontingent identity theses.

[13] This observation was first made by Ramsey in his adoption of logical atomism. Cf. Ramsey 1960.

[14] Cf. Cocchiarella, 1975, Section 5.

[15] Cf. Carnap, 1946, p.37 and 1947, Section 40. Unlike Carnap, Barcan assumed the formula as an axiom, and gave no explanation of why it should be accepted.

[16] Carnap 1947, p. 37.

3 The Primary Semantics of Logical Necessity

Let us now turn to what we take to be the primary semantics of logical necessity.[17] Our terminology will proceed as a natural extension of the syntax and semantics of standard first-order logic with identity. A formal language consists then just of predicate constants (of various finite degree) and objectual constants.

As **primitive logical constants** we take:

$\rightarrow$ the material conditional sign,

$\neg$ the negation sign,

$\forall$ the universal quantifier,

$=$ the identity sign, and

$\Box$ the logical necessity sign.

The conjunction, disjunction, biconditional, existential quantifier and possibility signs—$\wedge$, $\vee$, $\leftrightarrow$, $\exists$ and $\Diamond$, respectively—are understood to be defined in the usual way as metalinguistic abbreviatory devices. The formulas of a language are defined as in the logic of actual and possible objects, except that now the quantifier $\forall^e$ for existent objects is excluded and the logical necessity operator $\Box$ is included.

Because all possible objects are actual objects in logical atomism, we can restrict the notion of an model suited for a formal language L as follows.

Definition: If L is a language, then a **model** $\mathfrak{A}$ *is an L-model* if, and only if, for some nonempty set D and function R on L, $\mathfrak{A} = \langle D, R\rangle$, and for each objectual constant $a \in L$, $R(a) \in D$ and for each positive integer n and each n-place predicate $F^n \in L$, $R(F^n) \subseteq D^n$, i.e., $R(F^n)$ is a set of n-tuples of members of D.

Note: In the context of logical atomism, a model $\langle D, R\rangle$ for a language L represents *a possible world of a logical space* based upon D as the universe of objects of that space and L as the predicates characterizing the *atomic states of affairs* of that space. The possible worlds of this logical space are the L-models having D as their domain and that assign to each objectual constant a in L the same denotation that R assigns to a, because a is a "rigid designator".

Definition: If $\mathfrak{A} = \langle D, R\rangle$ is a model for a language L representing the actual world, then *the logical space determined by* $\mathfrak{A}$ = *the totality of possible worlds based on* $\mathfrak{A}$, in symbols $Wlds(\mathfrak{A})$, is defined as follows:
$Wlds(\mathfrak{A}) = \{\langle D,R'\rangle : \langle D,R'\rangle$ is an L-model and for all objectual constants a in L, $R'(a) = R(a)\}$.

[17] One or another version of this primary semantics for logical necessity, it should be noted, occurs in Carnap, 1946; Kanger, 1957; Beth, 1960 and Montague, 1960. Only Carnap, however, was clear about the association of his semantics with logical atomism.

Definition: A *is an assignment* (of values to variables) in a domain D if, and only if, A is a function with the set of objectual variables as domain and such that for each variable x, $A(x) \in D$.

Definition: If A is an assignment in a domain D, x is an objectual variable and $d \in D$, then $A(d/x)$ is an assignment in D that is exactly like A except for its assigning d to x.

Definition: If L is a language, $\mathfrak{A} = \langle D, R \rangle$ is an L-model, A is an assignment in D, and a is a variable or an individual constant in L, then (the denotation of a in $\mathfrak{A}$):

$$den_{\mathfrak{A},A} = \begin{cases} A(a) \text{ if } a \text{ is a variable} \\ R(a) \text{ if } a \text{ is an objectual constant} \end{cases}.$$

The **satisfaction in** $\mathfrak{A}$ of a formula φ of L by an assignment A in D, in symbols $\mathfrak{A}, A \models \varphi$, is recursively defined as follows:

1. $\mathfrak{A}, A \models (a = b)$ iff $den_{\mathfrak{A},A}(a) = den_{\mathfrak{A},A}(b)$;
2. $\mathfrak{A}, A \models F^n(den_{\mathfrak{A},A}(a_1, ..., a_n)$ iff $\langle den_{\mathfrak{A},A}(a_1), ..., den_{\mathfrak{A},A}(a_n)\rangle \in R(F^n)$;
3. $\mathfrak{A}, A \models \neg\varphi$ iff $\mathfrak{A}, A \not\models \varphi$;
4. $\mathfrak{A}, A \models (\varphi \rightarrow \psi)$ iff either $\mathfrak{A}, A \not\models \varphi$ or $\mathfrak{A}, A \models \psi$;
5. $\mathfrak{A}, A \models (\forall x)\varphi$ iff for all $d \in D$, $\mathfrak{A}, A(d/x) \models \varphi$; and
6. $\mathfrak{A}, A \models \Box\varphi$ iff for all $\mathfrak{B} \in Wlds(A)$, $\mathfrak{B}, A \vDash \varphi$.

The *truth* of a formula in a model (indexed by a language suitable to that formula) is as usual the satisfaction of the formula by every assignment in the universe of the model. *Logical truth* is then truth in every model (indexed by any appropriate language).

Definition: If L is a language, φ is a standard formula of L, $\mathfrak{A} = \langle D, R \rangle$, and $\mathfrak{A}$ is an L-model, then φ is true in $\mathfrak{A}$ if, and only if, for each assignment A in D, $\mathfrak{A}, A \models \varphi$.

Definition: φ *is logically true* if, and only if, for some language L, φ is a formula of L, and φ is true in every L-model.

These definitions are the natural extensions of the same semantical concepts as defined for the modal free formulas of standard first-order predicate logic with identity.

> Note that every model, because it specifies both a domain and a language, determines both a unique logical space and a possible world of that space. In this regard, the clause for the necessity operator in the above definition of satisfaction is the natural extension of the standard definition of satisfaction and interprets the necessity operator as ranging over *all* the possible worlds (models) of the logical space to which the given one belongs.

4 The Modal Thesis of Anti-Essentialism

Now it may be objected that logical atomism is an inappropriate framework upon which to base a system of quantified modal logic; for if any framework is a paradigm of anti-essentialism, it is logical atomism. The objection is void and begs the question, because it assumes that any system of quantified modal logic is committed to essentialism insofar as it allows quantifiers to reach into modal contexts, i.e., insofar as it allows *de re* modalities.

> Indeed, to the contrary, the above semantics for logical atomism provides the clearest example of why quantified modal logic does not automatically commit one to any non-trivial form of essentialism; and that is because in this semantics we can actually validate anti-essentialism in the form of a modal thesis, as Rudolf Carnap was the first to point out in terms of his equivalent state-description semantics.[18]

The general idea of the modal thesis of anti-essentialism is that if a predicate expression or open formula φ in which no objectual constants occur *can be true of* some objects in a given universe (satisfying a given identity-difference condition with respect to the variables free in φ), then φ *can be true of* any objects in that universe (satisfying the same identity-difference conditions).

> In other words, no condition is essential to some objects that is not essential to all, which is as it should be if necessity means logical necessity.[19]

The restriction to identity-difference conditions mentioned above can be dropped, it should be noted, if nested quantifiers are interpreted exclusively and not, as we have done, inclusively where, e.g., it is allowed that the value of y in $(\forall x)(\exists y)\varphi(x,y)$ can be the same as the value of x, as for example in $(\forall x)(\exists y)(x = y)$.[20] Now our point is that when nested quantifiers are interpreted exclusively, then identity and difference formulas are superfluous—which is especially appropriate in logical atomism where an identity formula does not represent an atomic state of affairs.[21]

Retaining the inclusive interpretation and identity as primitive, however, an *identity-difference condition* is defined as follows.

Definition: If $x_1, ..., x_n$ are distinct objectual variables, then *an identity-difference condition for* $x_1, ..., x_n$ is a conjunction of one each but not both of the formulas $(x_i = x_j)$ or $(x_i \neq x_j)$, for all i, j such that $1 \leq i < j \leq n$.

[18] See Carnap, 1946, T10-3.c, p.56, for the first formulation of this thesis ever to be given, and also Parsons, 1969 for a much later formulation.

[19] If individual constants do occur in a formula, they can be replaced uniformly by distinct new individual variables not already occurring in the formula.

[20] See Hintikka, 1956 for a development of the exclusive interpretation.

[21] Cf. Wittgenstein, 1961, and Cocchiarella, 1987, chapter V1.

Note: Because there are only a finite number of nonequivalent identity-difference conditions for $x_1, ..., x_n$, we can assume an ordering, $ID_1(x_1, ..., x_n)$,..., $ID_j(x_1, ..., x_n)$, is given of all of these nonequivalent conditions.

The modal thesis of anti-essentialism may now be stated for all formulas φ in which no objectual constants occur as follows: for all positive integers j such that $1 \leq j \leq n$, every formula of the form,

$$(\exists x_1)...(\exists x_n)[IDj(x_1, ..., x_n) \wedge \Box\varphi] \rightarrow (\forall x_1)...(\forall x_n)[IDj(x_1, ..., x_n) \rightarrow \Box\varphi]$$

is to be logically true, where $x_1, ..., x_n$ are all the distinct objectual variables occurring free in φ. We can also phrase this thesis in terms of its equivalent contrapositive form:

$$(\exists x_1)...(\exists x_n)[IDj(x_1, ..., x_n) \wedge \Diamond\varphi] \rightarrow (\forall x_1)...(\forall x_n)[IDj(x_1, ..., x_n) \rightarrow \Diamond\varphi]$$

Note: Where $n = 0$, the above formula is understood to be just $(\Box\varphi \rightarrow \Box\varphi)$; and where $n = 1$, it is understood to be just $(\exists x)\Box\varphi \rightarrow (\forall x)\Box\varphi$, or equivalently $(\exists x)\Diamond\varphi \rightarrow (\forall x)\Diamond\varphi$.

The validation of the thesis in our present semantics is easily seen to be a consequence of the following lemma (whose proof is by a simple induction on the formulas of L). That is, given that some objects satisfy φ in some L-model, then, by the following lemma, any permutation of those objects with any others in a domain D of the same size will also satisfy φ in some other L-model with that domain D, i.e., there will be an isomorphism between the two L-models.

LEMMA: If L is a language, $\mathfrak{A}, \mathfrak{B}$ are L-models, and h is an isomorphism of $\mathfrak{A}$ with $\mathfrak{B}$, then for all formulas φ of L and all assignments A in the universe of $\mathfrak{A}$, $\mathfrak{A}, A \models \varphi$ if, and only if, $\mathfrak{B}, A/h \models \varphi$.[22]

As already noted, one of the consequences of the modal thesis of anti-essentialism is the reduction of all *de re* formulas to *de dicto* formulas. Such a consequence indicates the correctness of our association of the present semantics with logical atomism.

Note: A *de re* formula φ is one in which some objectual variable has a free occurrence in a subformula of φ of the form $\Box\psi$, and hence a variable that can be bound by a quantifier applied to φ. A *de dicto* formula is a formula that is not *de re*.

[22] We understand h to be an isomorphism of $\mathfrak{A}$ with $\mathfrak{B}$ if (1) h is a 1-1 mapping of the domain of $\mathfrak{A}$ onto the domain of $\mathfrak{B}$, (2) for each individual constant $a \in L$, $den_{\mathfrak{B},A/h}(a) = h(den_{\mathfrak{A},A}(a))$, and (3)for all positive intergers n and n-place predicate constants $F \in L$, the extension F is assigned in B is $\{\langle h(d_{1)}, ..., h(d_n)\rangle$: for $\langle d_1, ...d_n\rangle$ in the extension that F is assigned in $\mathfrak{A}\}$. Also, we take the relative product A/h to be that assignement in the domain of $\mathfrak{B}$ such that for all variables x, $A/h(x) = h(A(x))$. In the case of an atomic formula in the inductive argument for this lemma, we have $\mathfrak{A}, A \models F(a_1, ..., a_n)$ iff $\langle den_{\mathfrak{A},A}(a_1), ..., den_{\mathfrak{A},A}(a_n)\rangle \in R(F)$ iff $\langle h(den_{\mathfrak{A},A}(a_1)), ..., h(den_{\mathfrak{A},A}(a_n))\rangle \in R'(F)$, where R, R' are the assignments to predicate constants in $\mathfrak{A}$ and $\mathfrak{B}$, respectively; and therefore $\mathfrak{A}, A \models F(a_1, ..., a_n)$ iff $\mathfrak{B}, A/h \models F(a_1, ..., a_n)$. The remaining cases follow in each case by the inductive hypothesis.

De Re **Elimination Theorem:** For each *de re* formula φ, there is a *de dicto* formula ψ such that $(\varphi \leftrightarrow \psi)$ is logically true.[23]

5 An Incompleteness Theorem for the Primary Semantics

Another result of the semantics for logical atomism is its *essential incompleteness* with respect to any language containing at least one relational predicate. This result depends on the conditional possibility of there being infinitely many objects, and a relational predicate is needed in order to state such an infinitary condition. In other words, an essential incompleteness theorem results if there are relational states of affairs in logical atomism.

> If there are only properties, i.e., monadic states of affairs, then the formal ontology is not only complete but decidable as well. The above semantics yields both a completeness theorem and a decision procedure for logical truth, in other words, for any language containing only monadic predicates.

The incompleteness theorem for languages containing a relational predicate is easily seen to follow from the following lemma and the well-known fact that the modal-free nonlogical truths of a first-order language containing at least one relational predicate is not recursively enumerable.[24]

LEMMA: If ψ is a sentence that is satisfiable, but only in an infinite model, and φ is a modal-free and identity-free sentence and φ, ψ contain no objectual constants, then $(\psi \to \neg\Box\varphi)$ is logically true iff φ is not logically true.[25]

[23] A proof of this theorem can be found in [McKay, 1975]. Briefly, where $x_1, ..., x_n$ are all the distinct individual variables occurring free in φ and $ID_1(x_1, ..., x_n), ..., D_k(x_1, ..., x_n)$ are all the nonequivalent identity-difference conditions for $x_1, ..., x_n$, then the equivalence in question can be shown if ψ is obtained from φ by replacing each subwff $\Box\chi$ of φ by:

$$\begin{array}{lcl} [ID_1(x_1, ...x_n) \wedge \Box\forall x_1...\forall x_n(ID_1(x_1, ..., x_n) & \to & \chi)] \vee ... \\ \vee[ID_k(x_1, ..., x_n) \wedge \Box\forall x_1...\forall x_n(ID_k(x1, ..., xn) & \to & \chi)]. \end{array}$$

[24] Cf. Cocchiarella, 1975.

[25] **Proof.** Assume the antecedent and, for the left-to-right direction that $(\psi \to \neg\Box\varphi)$ is logically true. We note that if φ were logically true, then it would be true in every L-model for any language L of which φ is a formula; but then φ would be true in an infinite L-model $\mathfrak{A}$ in which ψ is satisfiable, in which case, by assumption, $\neg\Box\varphi$ would be true $\mathfrak{A}$ as well; but that is impossible because φ would then be false in some L-model when by assumption φ is logically true, and therefore true in every L-model. For the right-to-left direction, suppose φ is not logically true. Let $\mathfrak{A}$ be an arbitrary L-model for any language L of which φ and ψ are formulas. It suffices to show that $(\psi \to \neg\Box\varphi)$ is true in $\mathfrak{A}$. If ψ is not satisfiable in $\mathfrak{A}$, then $(\psi \to \neg\Box\varphi)$ is vacuously true in $\mathfrak{A}$. Suppose then that ψ is satisfiable in $\mathfrak{A}$. Then, by

THEOREM: If L is a language containing at least one relational predicate, then the set of formulas of L that are logically true is not recursively enumerable.

Proof. Because the modal and identity free formulas of L that are not logically true are not recursively enumerable, it follows by the above lemma that the logically true formulas of L of the form $(\psi \to \neg\Box\varphi)$ are also not recursively enumerable, and hence that the set of formulas of L that are logically true is not recursively enumerable. ■

This last result does not affect the association we have made of the primary semantics with logical atomism. Indeed, given the Löwenheim-Skolem theorem, what the above lemma shows is that

> there is a complete concurrence between logical necessity as an internal condition of modal free propositions, or of their corresponding states of affairs, and logical truth as a semantical condition of the modal free sentences expressing those propositions, or representing their corresponding states of affairs. And that of course is as it should be if the operator for logical necessity is to have only formal and no material content.

Finally, it should be noted that the above incompleteness theorem explains why Carnap was not able to prove the completeness of the system of quantified modal logic formulated in Carnap, 1946. For on the assumption that the number of objects in the universe is denumerably infinite, Carnap's state description semantics is essentially that of the primary semantics restricted to denumerably infinite models; and, of course, precisely because the models are denumerably infinite, the above incompleteness theorem applies to Carnap's formulation as well. Thus, the reason why Carnap was unable to carry though his proof of completeness is finally answered.

6 The Secondary Semantics of Metaphysical Necessity

Like the situation in standard second-order logic, the incompleteness of the primary semantics can be avoided by allowing the quantification over possible worlds in the interpretation of necessity to refer not to all of the possible worlds (models) of a given logical space but only to those in a given non-empty set of such.

hypothesis, D, the domain of $\mathfrak{A}$, is infinite. Now because φ is modal and identity free and not logically true, then, by Löwenheim-Skolem theorem, φ must be false in some L-model $\mathfrak{B}$ having D as its domain, and hence, because no objectual constants occur in φ or ψ, $\mathfrak{B} \in Wlds(\mathfrak{A})$. But then, by the semantic clause for $\Box$, $\neg\Box\varphi$ is true in $\mathfrak{A}$, and therefore so is $(\psi \to \neg\Box\varphi)$. ■

> That is, by allowing a "cut-down" on the notion of "all possible worlds" in the interpretation of necessity, we can obtain a completeness, instead of an incompleteness, theorem. Of course, the world in question must be in the "cut-down" as part of the definition of satisfaction.

Accordingly, where L is a language and D is a non-empty set, we understand *a model structure based* on **D** and **L** to be a pair $\langle\mathfrak{A},K\rangle$, where $\mathfrak{A} \in K$, K is a set of L-models all having D as their domain of discourse and all of the objectual constants in L are assigned the same denotation in each L-model in K.

Definition: If L is a language and D is a nonempty set, then $\langle\mathfrak{A},K\rangle$ *is a model structure based on* **D** and **L** if, and only if, $\mathfrak{A} \in K$ and K is a set of L-models all having D as their domain of discourse and all agreeing on the assignment of members of D to the objectual constants in L, i.e., the assignment of a members of D to objectual constants in L is the same for each member of K.

The satisfaction of a formula φ of L in such a model structure by an assignment A in D, in symbols $\langle\mathfrak{A},K\rangle, A \models \varphi$, is recursively defined exactly as in the primary semantics *except for clause (6)*, which is now defined as follows:

6. $\langle\mathfrak{A},K\rangle, A \models \Box\varphi$ iff for all $\mathfrak{B} \in K$, $\langle\mathfrak{B},K\rangle, A \models \varphi$.

Instead of logical truth, a formula is understood to be **universally valid** if it is satisfied by every assignment in every model structure based on a language to which the formula belongs.

Definition: φ *is universally valid* if, and only if, for every language L, every nonempty domain D, every model structure $\langle\mathfrak{A},K\rangle$ based on D and L, and every assignment A in $\mathfrak{A}$, if φ is a formula of L, then $\langle\mathfrak{A},K\rangle, A \models \varphi$.

Where **QS5** is standard first-order logic with identity supplemented with the axioms of **S5** propositional modal logic, a completeness theorem for the secondary semantics of logical necessity was proved by Kripke in 1959.

Completeness Theorem: A set Γ of formulas is consistent in **QS5** if, and only if, all the members of Γ are simultaneously satisfiable in a model structure; and (therefore) a formula φ is a theorem of **QS5** if, and only if, φ is universally valid.

Despite the above completeness theorem, the secondary semantics has too high a price to pay as far as logical atomism is concerned.

> Unlike the situation in the primary semantics, the secondary semantics does not validate the modal thesis of anti-essentialism—i.e., it is false that every instance of the thesis is universally valid.

Example 1 *As an example of a false instance of the thesis, consider a model structure* $\langle\mathfrak{A}, K\rangle$*, where* $K = \{\mathfrak{A}, \mathfrak{B}\}$, $D = \{d_1, d_2\}$, $d_1 \neq d_2$*, and* F *is a monadic predicate that is assigned only* d_1 *in* $\mathfrak{B}$ *and nothing in* $\mathfrak{A}$*. Then* $(\exists x)\Diamond F(x)$ *is true in* $\langle\mathfrak{A}, K\rangle$*, whereas* $(\forall x)\Diamond F(x)$ *is false in* $\langle\mathfrak{A}, K\rangle$*, which invalidates the modal thesis of anti-essentialism (in the case of a monadic formula).*

The reason why the modal thesis of anti-essentialism can be invalidated in the secondary semantics is because necessity no longer represents an invariance through all the possible worlds of a given logical space but only through those in a nonempty set of such worlds.

> In this way, necessity is no longer a purely formal concept having no material content the way it is in logical atomism. Instead, necessity is now allowed to represent *an internal condition* of states of affairs—i.e., a condition that has material and not merely formal content—for what is invariant through all of the members of such a nonempty set need not be invariant though all the possible worlds (models) of the logical space to which those in the set belong.

One example of how such material content affects the implicit metaphysical background can be found in monadic modal predicate logic. First let us note a well-known fact about modal-free monadic predicate logic.

Note: Modal-free monadic predicate logic is decidable and no modal-free monadic formula can be true in an infinite model unless it is true in a finite model as well. Therefore, any substitution instance of a modal-free monadic formula for a relational predicate in an axiom of infinity—i.e., a formula that is true only in an infinite domain of discourse—is not only false but logically false; and hence, contrary to certain metaphysical views there can be no modal-free analysis or reduction of all relational predicates (or open formulas with two or more free variables) in terms only of monadic predicates, i.e., in terms only of modal-free monadic formulas.

Now the same result also holds for quantified monadic modal logic with respect to the primary semantics, where "all possible worlds" means *all possible worlds.*

Theorem: Modal monadic predicate logic is also decidable with respect to the primary semantics of logical atomism; and therefore no monadic formula, modal-free or otherwise, can be true in an infinite model unless it is also true in a finite model.[26] That is, there can be no reduction of all relational predicates or open formulas in terms only of monadic formulas, modal free or otherwise.

[26] Cf. Cocchiarella, 1975. This is proved by interpreting $\Box$ as a string of universal quantifiers on the predicates occurring within the scope of $\Box$, and thereby translating modal formulas into modal-free formulas of second-order monadic predicate logic, which is known to be decidable.

The following formula, e.g., is true in some models based on an infinite domain but false in all models based on a finite domain. This formula, in other words, can be taken as an axiom of infinity.

$$(\forall x)\neg R(x,x) \wedge (\forall x)(\exists y)R(x,y) \wedge (\forall x)(\forall y)(\forall z)[R(x,y) \wedge R(y,z) \to R(x,z)]$$

Consider substituting the open formula (with two free variables) $\Diamond[F(x) \wedge G(y)]$, with two monadic predicate constants F and G, for the two-place predicate R in this formula. The first conjunct then yields the following substitution instance.

$$(\forall x)\neg\Diamond[F(x) \wedge G(x)],$$

which is equivalent to

$$\neg\Diamond(\exists x)[F(x) \wedge G(x)].$$

But the formula $\Diamond(\exists x)[F(x) \wedge G(x)]$ is logically true in the primary semantics of logical atomism, and hence its negation is logically false, which shows that that the conjunction that is an instance of the above infinity formula is logically false.

In other words, where L is any language having the monadic predicates F and G as members, then given any nonempty domain D there will be an L-model $\mathfrak{A}$ in which $[F(x) \wedge G(x)]$ is satisfied by some member of D, and hence $(\exists x)[F(x) \wedge G(x)]$ will be true in $\mathfrak{A}$, which means that $\Diamond(\exists x)[F(x) \wedge G(x)]$ will be true in any world (L-model) in the logical space determined by $\mathfrak{A}$, i.e., true in any world in $Wlds(\mathfrak{A})$, and hence that $\Diamond(\exists x)[F(x) \wedge G(x)]$ is logically true with respect to the primary semantics of logical atomism. Thus, substituting $\Diamond[F(x) \wedge G(y)]$ for R in the above formula representing an axiom of infinity results in a logically false sentence in the primary semantics.

With respect to the secondary semantics, however, the situation is quite different, because all we need do is exclude all of the L-models in the logical space based on L and D in which $(\exists x)[F(x) \wedge G(x)]$ is true. The "cut-down" or resulting model structure $\langle \mathfrak{B}, K \rangle$ will be such that $\neg\Diamond(\exists x)[F(x) \wedge G(x)]$ is true in all of the models in K. Equivalently, the formula

$$\Box(\forall x)[F(x) \to \neg G(x)],$$

which clearly represents a necessary, i.e., an internal, relation between being an F and not-being a G, will be true in $\langle \mathfrak{B}, K \rangle$ with respect to the secondary semantics.

Instead of modal monadic predicate logic being decidable the way it is in the primary semantics, modal monadic predicate logic is undecidable in the secondary semantics, as Kripke has shown. Moreover, on the basis of that semantics a modal analysis of relational predicates in terms of monadic[27] predicates can in general be given. Indeed, substituting $\Diamond[F(x) \wedge G(y)]$ for the binary predicate R in the above infinity axiom results in a modal monadic sentence that is true in some model structure based on an infinite universe and false in all model structures based on a finite domain.

[27] Cf. Kripke 1962.

> Somehow, in other words, by means of a "cut-down" on the notion of "all possible worlds", relational content has been incorporated into the semantics for necessity, and thereby of possibility as well.

In this respect, the secondary semantics is *not* the semantics of a merely formal, or logical, necessity but of a necessity having material, nonlogical content as well.

7 Concluding Remarks

Kripke does not himself refer to the necessity of his semantics as a formal, or logical, necessity, but as a metaphysical necessity. He has also argued that not every necessary proposition is *a priori*, and that not every *a posteriori* proposition is contingent.[28]

> It is because Kripke is concerned with a metaphysical necessity and not a logical, or formal, necessity that not every necessary proposition needs to be *a priori*, nor every *a posteriori* proposition contingent.
>
> But such a position cannot be taken as a refutation of the claim in logical atomism that every logically necessary proposition is *a priori* and that every *a posteriori* proposition is logically contingent.
>
> These are two different metaphysical frameworks, each with its own notion of necessity and thereby of contingency as well.

Now we can extend the notion of a model structure $\langle \mathfrak{B}, K \rangle$ based on a domain D so that instead of having D represent the same universe of existing objects in all of the worlds in K, D would represent only the same domain of possible objects of the structure $\langle \mathfrak{B}, K \rangle$, i.e., the union, or sum, of all of the objects that exist in some world or other in K. We would then distinguish a universe of *existing* objects for each world $\mathfrak{A} \in K$, i.e., the objects that exist in $\mathfrak{A}$ and not necessarily in the other worlds in K, just as we did in the semantics for the logic of actual and possible objects in tense logic. We would then reintroduce the actualist quantifiers $\forall^e$ and $\exists^e$ and the free logic of actualism to represent the restricted quantification over existing objects. We could either retain the full logic of actual and possible objects in that case, or we could restrict ourselves to just the actualist modal logic, depending on whether we want to represent modal possibilism or modal actualism.[29]

We can also deepen the material content of the secondary semantics by adding a relation $\mathcal{R}$ of accessibility between possible worlds, i.e., between the models in K, where $\langle \mathfrak{B}, K \rangle$ is a model structure, so that necessity in a world

[28] Cf. Kripke, 1971, p. 150.
[29] See Cocchiarella 1984, section 6.

$\mathfrak{A} \in K$ is *an invariance condition that is restricted to the worlds in K that are accessible from* $\mathfrak{A}$.[30] If $\mathcal{R}$ is an equivalence relation, then the modal logic **S5**, whether as actualist or possibilist, characterizes the class of model structures that result. Other quantified modal logics characterize classes of model structures in which the accessibility relation between models is weaker than an equivalence relation. The modal logic **S4**, for example, characterizes the class of model structures in which the accessibility relation is transitive and reflexive.

The question remains, however, as to just what ontology such a secondary semantics represents, and in particular what notion of necessity other than logical necessity is in question. Calling the result a metaphysical modality is not an adequate answer. We need a philosophical account of what principles determine the "cut-down" on possible worlds (models), and how the accessibility relation between worlds is to be explained in terms of such principles. A set-theoretic structure with respect to which a completeness theorem can be proved is not itself such a philosophical account.

In the ontology of conceptual realism, for example, the "cut-down" on possible worlds can be accounted for either in terms of tense logic or in terms of the network of natural laws that are part of nature's causal matrix. Not all logically possible states of affairs will be realized in time, so that time itself provides a metaphysical ground for such a "cut-down". That is why the Aristotelian and Diodorean modalities definable in terms of a local time are unproblematic. The earlier-than relation of a local time, or the signal relation of a causally connected system of local times also provide unproblematic metaphysical bases for different accessibility relations, as well as ontological grounds for different modal logic.

Similarly, an ontological ground for such a "cut-down" on all logically possible worlds can be given in terms of the set $\mathcal{L}$ of laws of nature. Thus, for example, where K is a set of the possible worlds in which all of the laws in $\mathcal{L}$are true, and $\mathbf{B} \in K$, the model structure $\langle \mathbf{B}, K \rangle$ would characterize an invariance condition based on the laws in $\mathcal{L}$. Different sets of laws would then determine different model structures. But because each model structure $\langle \mathfrak{B}, K \rangle$ would be determined by the same set of laws, then all of the worlds in K would have the same laws of nature, and hence each would be accessible from every other member of K. The modal logic that results would then be **S5**.

If the laws of nature are characterized in terms of underlying causal mechanisms, then, because the causal relation is transitive, but not also symmetric, the modal logic that results would be **S4** instead of **S5**. If the causal mechanisms in nature are themselves based on a hierarchy of natural kinds, then the model structures must be restricted to those in which the axioms of the logic of natural kinds are true as well. It is by these kinds of constraints that we give content to what is meant by a "cut-down" on the notion of all possible worlds, and thereby on the notions of necessity and possibility.

> The real problem of quantified modal logic for an ontology other than logical atomism, we have said, is to give an

[30] Ibid., section 7.

ontological account of the "cut-down" on the notion of "all possible worlds" and of the accessibility relation between possible worlds.

We have indicated how such an account can be given for conceptual realism in terms of time, or the laws of nature, or the causal mechanisms of a hierarchically structured universe of the natural kinds that make up the causal nexus of the world.

The question is: can this really be done other than in the modalities constructible within tense logic on the basis of time or the causal or natural modalities of Aristotelian essentialism?

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