Formal Theories of Predication, Part I: Lecture Three Intensive Course STOQ 50547 II Semester April 26–30, 2004

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1 Introduction

A formal ontology, we have noted, is based on a theory of predication and not on set theory as a theory of membership. Set theory, of course, might be used as a model-theoretic guide in the construction of a theory of predication; but such a guide can be misleading if we confuse membership with predication. A theory of predication, as we have noted, depends on what theory of universals it is designed to represent, where, by a universal we mean that which can be predicated of things. A universal is not just an abstract entity, in other words, but something that has a predicative nature, which sets do not have.

Our methodology in studying a theory of predication is to reconstruct it as a second-order predicate logic that represents the salient features of that theory. Now by a second-order predicate logic we mean an extension of first-order predicate logic in which quantifiers are allowed to reach into the positions that predicates occupy in formulas, as well as into the subject or argument positions of those predicates.[1] This means that just as the quantifiers of first-order logic are indexed by object variables, which are said to be bound by those quantifiers, so too the quantifiers of second-order logic are indexed by predicate variables, which are said to be bound by those quantifiers. In this respect we need only add n-place predicate variables, for each natural number n, to our syntax for first-order logic. We will use for this purpose the capital letters F^n, G^n, H^n, with or without numerical subscripts, as n-place predicate variables; but we will generally drop the superscript when the context makes clear the degree of the predicate variable. Sometimes, for relational predicates, i.e., where $n > 1$, we will also use R and S as relational predicate variables as well.

[1] Here, by first-order predicate logic we mean the logic of possible objects described in lecture 2.

We still understand a formal language L to be a set of object and predicate constants. The atomic formulas of such a language L are defined in the same way as in first-order logic, except that now the predicate of an atomic formula might be a predicate variable instead of a predicate constant. Thus, the (second-order) formulas of a language L are understood to be defined as follows.

Definition: χ is **a (second-order) formula** of a language L if, and only if, for all sets K, if (1) every atomic formula of L is in K, and (2) for all $\varphi, \psi \in K$, all object variables x, and, for each natural number n, all n-place predicate variables F^n, $\neg\varphi$, $(\varphi \to \psi)$, $(\forall x)\varphi$, and $(\forall F^n)\varphi \in K$.

It is formally convenient, incidentally, to take *propositional variables* to be 0-place predicate variables, and hence to allow for atomic formulas of the form F^0. In other words, a propositional variable is a predicate variable that takes zero many terms as arguments to result on an atomic formula. We will generally use the capital letters P, Q, with or without numerical subscripts as propositional variables.

A principle of induction over second-order formulas follows of course just as it did in first-order logic. By way of axioms, we add the following to the ten we already gave for standard first-order logic. These axioms are the distribution law for predicate quantifiers, and the law of vacuous quantification:

(A11) $(\forall F^n)[\varphi \to \psi] \to [(\forall x)\varphi \to (\forall x)\psi]$

(A12) $\varphi \to (\forall F^n)\varphi$, where F^n is not free in φ

We also add the inference rule of universal generalization for predicate quantifiers:

UG$_2$: If $\vdash \varphi$, then $\vdash (\forall F^n)\varphi$.

2 Logical Realism

The axioms described so far apply to nominalism and conceptualism, as well as to logical realism. What distinguishes logical realism is an axiom schema that we call a *comprehension principle*, **(CP)**. This principle is stated as follows:

$$(\exists F^n)(\forall x_1)...(\forall x_n)[F(x_1, ..., x_n) \leftrightarrow \varphi], \qquad (\mathbf{CP})$$

where φ is a (second-order) formula in which F^n does not occur free, and $x_1, ..., x_n$ are pairwise distinct object variables occurring free in φ.

What the comprehension principle does in logical realism is posit the existence of a universal corresponding to every second-order formula φ. This is so even when φ is a contradictory formula, e.g., $\neg[G(x) \to G(x)]$. In other words, contrary to what has sometimes been held in the history of philosophy, there are properties and relations, that on logical grounds alone cannot have any instances. Such properties and relations cannot be excluded without seriously affecting the logic of logical realism. In particular, because we cannot effectively decide even when a first-order formula is contradictory, the logic would then not be recursively axiomatizable.

Note that by a simple inductive argument on the structure of the formula φ, a second-order analogue of Leibniz's law is provable independently of **(CP)**:

$$(\forall x_1)...(\forall x_n)[F(x_1,...,x_n) \leftrightarrow \varphi] \rightarrow (\psi \leftrightarrow \psi[\varphi/F(x_1,...,x_n)]),$$

where $\psi[\varphi/F(x_1,...,x_n)]$ is the result of properly substituting φ for the free occurrences of F in ψ with respect to the object variables $x_1,...,x_n$.[2] From this, by $\mathbf{UG}_2$, axioms **(A11)**, **(A12)** and elementary transformations, what follows as a theorem schema is the law of universal instantiation of formulas for predicate variables:

$$(\forall F)\psi \rightarrow \psi[\varphi/F(x_1,...,x_n)], \qquad (\mathrm{UI}_2)$$

where φ can be properly substituted for F in ψ. The contrapositive of $(\mathbf{UI}_2)$ is of course the second-order law for existential generalization,

$$\psi[\varphi/F(x_1,...,x_n)] \rightarrow (\exists F)\psi. \qquad (\mathrm{EG}_2)$$

The second-order predicate logic described so far, where the comprehension principle **(CP)** is valid for any formula φ, is sometimes called "standard" second-order logic. The use of "standard" in this context should not be confused with the notion of a "standard" set-theoretic semantics for second-order logic, i.e., a semantics based on confusing predication with membership in a set D, where *all* sets of n-tuples drawn from the power-set of D^n are taken as the values of the n-place predicate variables.[3] Second-order predicate logic, it is well-known, is essentially incomplete with respect to this so-called "standard set-theoretic semantics". But, as we have already noted (in our first lecture), whether or not that incompleteness applies to the theory of universals that is the basis of a second-order predicate logic is another matter altogether.

Now it is noteworthy that had we assumed $(\mathbf{UI}_2)$ as an axiom schema instead of **(CP)**, then, by $(\mathbf{EG}_2)$, the comprehension principle **(CP)** would be derivable as a theorem schema instead. This raises the question of whether or not there are any reasons to prefer **(CP)** over $(\mathbf{UI}_2)$, or $(\mathbf{UI}_2)$ over **(CP)**.

One reason to prefer $(\mathbf{UI}_2)$ over **(CP)** is that whereas **(CP)** is existential in form $(\mathbf{UI}_2)$ is universal. Gottlob Frege, who was the first to formulate a

[2] If $x_1,...x_n$ are pairwise distinct object variables, then $\psi[\varphi/F(x_1,...,x_n)]$ is just ψ unless the following two conditions are satisfied: (1) no free occurrence of F^n in ψ occurs with a subformula of ψ of the form $(\forall a)\chi$, where a is a predicate or object variable other than $x_1,...x_n$ that occurs free in φ; and (2) for all terms $a_1,...,a_n$, if $F(a_1,...,a_n)$ occurs in ψ in such a way that the occurrence of F^n in question is a free occurrence, then for each i such that $1 \leq i \leq n$, if a_i is a variable, then there is no subformula of φ of the form $(\forall a_i)\chi$ in which the variable x_i has a free occurrence. If these two conditions are satisfied, then $\psi[\varphi/F(x_1,...,x_n)]$ is the result of replacing, for all terms $a_1,...,a_n$, each occurrence of $F(a_1,...,a_n)$ in ψ at which F^n is free by an occurrence of $\varphi(a_1/x_1,...,a_n/x_n)$. When conditions (1) and (2) hold, we say that φ can be properly substituted for F^n in ψ.

[3] By D^n we understand the set of n-tuples drawn from D. In the so-called "standard" set-theoretic semantics for second-order logic, the power-set of D^n is taken as the range of values assigned to n-place predicate variables. The set theory involved here is assumed to be based on the iterative concept of set and Cantor's power-set theorem.

version of second-order logic, argued that the laws of logic should be universal in form, which is why he had a version of **(UI_2)** as one of his basic laws. Bertrand Russell assumed a version of **(EG_2)**, the contrapositve of **(UI_2)**, as a "primitive proposition" of his higher-order logic. This principle, according to Russell, "gives the only method of proving 'existence theorems'"[4], which suggests that Russell did not think that "existence theorems" of the form **(CP)** were provable in his logic.

It is elegant, perhaps, to have the basic laws of logic all be universal in form, but there is something to be said for putting our existential posits up front, and that is precisely what **(CP)** does—just as the related axiom **(A8)**, $(\exists x)(a = x)$, for first-order logic puts our our existential presuppositions up front for singular terms. The latter, i.e., axiom **(A8)**, may in fact be rejected, as we will see, once predicates are allowed to be transformed into singular terms as counterparts of abstract nouns. In other words, in a somewhat larger ontological context that we will consider later, where complex singular terms are allowed, certain of these complex singular terms will lead to a contradiction if axiom **(A8)**, which is the first-order counterpart of **(CP)**, is not modified along the lines of "free logic".

Can **(CP)** also be rejected in that larger ontological context? No, at least not in logical realism. In the ontology of natural realism, however, the assumption that a natural property or relation corresponds to any given predicate or formula is at best an empirical hypothesis. That is, just as whether or not a singular term denotes an object in free logic is not a question that can be settled by logical considerations alone, so too in natural realism the question whether or not a given predicate or open formula stands for a natural property or relation cannot be settled by logic alone. The comprehension principle **(CP)**, in other words, is not a valid thesis in natural realism. We will forego giving a fuller analysis of natural realism at this point, however, until a later lecture when we consider conceptual natural realism.

The status of the comprehension principle, **(CP)**, as these remarks indicate, is an important part of the question of what metaphysical theory of universals is being assumed as the basis of our logic as a formal theory of predication.

3 Nominalism

In nominalism, the basic thesis is that there are no universals, and that there is only predication in language. This suggests that the comprehension principle **(CP)** must be false in nominalism, which is why the formal theory of predication that is commonly associated with nominalism is standard first order predicate logic with identity. In fact, however, the situation is a bit more complicated than that.

It is true that according to nominalism first-order predicate logic gives a logically perspicuous representation of the predicative nature of the predicate expressions of language. It is the logico-grammatical roles that predicates have

[4] Russell & Whitehead 1910, p. 131.

in the logical forms of first-order predicate logic, in other words, that explains their predicative nature according to nominalism.

> Nominalism: The logico-grammatical roles that predicate expressions have in the logical forms of first-order predicate logic explains their predicative nature.

Predicate constants are of course assigned the paradigmatic roles in this explanation, but this does not mean that predicate constants are the only predicative expressions that must be accounted for in nominalism. In particular, any open first-order formula of a formal language L, relative to the free object variables occurring in that formula, can be used to define a new predicate constant of L.[5] Such an open formula would constitute the definiens of a possible definition for a predicate constant not already in that language. Accordingly, an open formula must be understood as implicitly representing a predicate expression of that formal language. Potentially, of course there are infinitely many such predicate constants that might be introduced into a formal language in this way, and some account must be given in nominalism of their predicative role.

> Question: how can nominalism, as a theory of predication, represent the predicative role of open first-order formulas.

Now an account is forthcoming by extending standard first order predicate logic to a second order predicate logic in which predicate quantifiers are interpreted substitutionally. That is, we can account for all of the nominalistically acceptable predicative expressions of an applied first-order language without actually introducing new predicate constants by simply turning to a second order predicate logic in which predicate quantifiers are interpreted substitutionally and where predicate variables have only first-order formulas as their substituents.[6] There are constraints that such an interpretation imposes, of course, and in fact, as we have shown elsewhere, those constraints are precisely those imposed on the comprehension principle in standard "predicative" second-order logic.[7] Here, it should be noted, the use of the word 'predicative' is based on Bertrand Russell's terminology in *Principia Mathematica,* where the restriction in question was a part of his theory of ramified types. Apparently, because of the liar and other semantical paradoxes, Russell, despite his being a logical realist at the time, thought that only the so-called "predicative" formulas should be taken as representing a property or relation.[8]

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[5] By *an open formula* we mean a formula in which some variables have a free occurrence.

[6] Strictly speaking predicate variables will have as subtituends any formula in which no *bound* predicate variable occurs, which means that free predicate variables are allowed to occur in such substituends.

[7] See Cocchiarella 1980.

[8] Russell's logical realism is most pronounced in his 1903 *Principles of Mathematics*. His later 1910 view in *Principia Mathematica* might more appropriately be described as a form of conceptual Platonism. From 1914 on, especially in his logical atomist phase, Russell is best described as a natural realist. See Cocchiarella 1991 for a description of Russell's higher-order logic as a form of conceptual Platonism.

The restriction, simply put, is that no formula containing bound predicate variables is to be allowed in the comprehension principle. The comprehension principle **(CP)**, in other words, is to be restricted as follows:

$$(\exists F^n)(\forall x_1)...(\forall x_n)[F(x_1,...,x_n) \leftrightarrow \varphi], \qquad \text{(CP!)}$$

where φ is a formula in which (1) no predicate variable has a bound occurrence, (2) F^n does not occur free in φ, and (3) $x_1,...,x_n$ are pairwise distinct object variables occurring free in φ.[9]

Under a substitutional interpretational the appearance of an existential posit regarding the existence of a universal in the quantifier prefix $(\exists F^n)$ is just that, an appearance and nothing more. In an applied formal language, this principle involves no ontological commitments under such an interpretation beyond those one is already committed to in the use of the first-order formulas of that language. That is, by interpreting predicate quantifiers substitutionally, **(CP!)** will not commit us ontologically to anything we are not already committed to in our use of first-order formulas, and, as we have said, it is the logico-grammatical role of predicate expressions in first-order logic that is the basis of nominalism's theory of predication.

Note that because the second-order analogue of Leibniz's law is provable independently of **(CP)**, it then is provable in our nominalistic logic just as it was in the logic for logical realism. But then, just as the universal instantiation law, **(UI$_2$)**, is provable in logical realism, we have a restricted version also provable in nominalism. That is, if no predicate variable has a bound occurrence in φ, then, by **(CP!)**, the following is provable in nominalism,

$$(\forall F)\psi \rightarrow \psi[\varphi/F], \qquad \textbf{(UI!}_2\textbf{)}$$

where φ can be properly substituted for F in ψ. From **(UI!$_2$)**, we can then derive the restricted version of existential generalization for predicates.

> What these observations indicate is that a comprehension principle can be used to make explicit what is definable in a given applied language, as well as to indicate, as in logical realism, what our existential posits are regarding universals.

Thus, where L is a formal language, P^n is an n-place predicate constant not in L, and ψ is a first-order formula in which $x_1,...,x_n$ are all of the distinct object variables occurring free, then

$$(\forall x_1)...(\forall x_n)[P(x_1,...,x_n) \leftrightarrow \psi]$$

is said to be *a possible definition in L of P*. Now it is just such a possible definition that is posited in **(CP!$_2$)**. In other words, as the definiens of such

[9] Russell used the exclamation mark as an way to indicate which formulas of his type theory were "predicative". We use it here as a way to indicate the relevant restriction on the comprehension principle.

a possible definition, the first-order formula ψ is implicitly understood to be a complex predicate of the language L. Of course, if the above were an explicit definition in L, then, by **(EG!$_2$)**, the relevant instance of **(CP!)** follows as provable in L.

The kind of definitions that are excluded in nominalism but allowed in logical realism are the so-called "impredicative" definitions; that is, those that in realist terms represent properties and relations that seem to presuppose a totality to which they belong. The definition of a least upper bound in real number theory is such a definition, for example, because, by definition, a least upper bound of a set of real numbers, is one of the upper bounds in that set. The exclusion of impredicative definitions is sometimes called the Poicaré-Russell vicious circle principle, because Henri Poincaré and Bertrand Russell were the first to recognize and characterize such a principle.[10]

4 Constructive Conceptualism

The notion of an "impredicative" definition is important in conceptualism as well as in nominalism, and it is basic to an important stage of concept-formation. Conceptualism, as we have noted, differs from nominalism in that it assumes that there are universals, namely concepts, that are the semantic grounds for the correct application of predicate expressions. Of course, conceptualism also differs from realism in that concepts are not assumed to exist independently of the human capacity for thought and concept-formation.

Conceptualism is a sociobiologically based theory of the human capacity for thought and concept-formation, and, more to the point, systematic concept-formation. Concepts themselves are types of cognitive capacities, and it is their exercise as such that underlies the speech and mental acts that constitutes our thoughts and communications with one another. But thought and communication exist only as coordinated activities that are systematically related to one another through the logical operations of thought; and it is with respect to the idealized closure of these operations that concept-formation is said to be systematic. It is only as a result of this closure, moreover, that the unity of thought as a field of internal cognitive activity is possible.

The coordination and closure of concepts does not occur all at once in the development of human thought, of course, nor is the structure of the closure the same at all stages in that development. In fact, the human child proceeds through stages of cognitive development that are of increasing structural complexity, corresponding in part to the increasing complexity of the child's developing brain. These stages, as Jean Piaget has noted, emerge as states of cognitive equilibrium with respect to certain regulatory processes that are constitutive of systematic concept-formation.[11] Different stages proceed as transformations from one state of cognitive equilibrium to another of increased structural complexity, where the need for such transformations arises from the child's inter-

[10] Cf. Poincaré 1906 and Russell 1906.
[11] Cp. Piaget 1977.

action with his environment and the tacit realization of the inadequacy of the earlier stages to understand certain aspects of the world of his experience. The later stages are states of "increasing re-equilibration" of the intellect, in other words, so that the result is an improved representation of the world.[12]

Now there is an important stage of cognitive equilibrium of logical operations that immediately precedes the construction of so-called "impredicative" concepts, which usually does not occur until post-adolescence. We refer to the logic of this stage as *constructive conceptualism*. The later, succeeding more mature stage at which "impredicative" concept-formation is realized is called *holistic conceptualism*, though we will generally refer to it later simply as conceptualism. It is in constructive conceptualism that "impredicative" definitions are excluded, and this exclusion occurs in the form that the comprehension principle takes in constructive conceptualism, which can be formally described as follows:

$$(\forall G_1)...(\forall G_k)(\exists F)(\forall x_1)...(\forall x_n)[F(x_1,...,x_n) \leftrightarrow \varphi], \qquad \text{(CCP!)}$$

where

(1) φ is a pure second-order formula, i.e., one in which no nonlogical constants occur,

(2) F is an n-place predicate variable such that neither it nor the identity sign occur in φ,

(3) φ is "predicative" in nominalism's purely grammatical sense, i.e., no predicate variable has a bound occurrence in φ,

(4) $G_1,...,G_k$ are all of the distinct predicate variables occurring (free) in φ, and

(5) $x_1,...,x_n$ are distinct object variables.

Now we should note that by the rule for universal generalization of quantifiers, **(UG)**, every instance of the conceptualist principle **(CCP!)** is derivable from the nominalist principle **(CP!).** But not every instance of the nominalist principle **(CP!)**, however, is also an instance of the conceptualist principle **(CCP!)**. In an applied formal language L, the formula φ in instances of **(CP!)** will in general be a first-order formula of L, and it may contain the identity sign and any predicate constant of L. Instances of **(CCP!)**, on the other hand, contain neither the identity sign nor any predicate constants of L.

Predicate constants are excluded from instances of **(CCP!)** because, unlike the situation in nominalism, a predicate constant (or first-order formula in terms of which such a constant might be defined) might not stand for a "predicative" concept, i.e., it might not stand for a value of the bound predicate variables. This is because the logic of predicate quantifiers in constructive conceptualism is like the logic of first-order quantifiers in free logic in that the logic is free of existential presuppositions regarding predicate constants, which means that a predicate constant must stand for a "predicative" concept in order to be a

[12] Ibid., p. 13 and §6 of chapter 1,

substituend of the bound predicate variables of the logic. The predicate quantifiers in nominalism, on the other hand, function like the objectual quantifiers of standard first-order logic; and that is because, as the paradigms of predication in nominalism, predicate constants do not differ from one another in their predicative role, which is why, under a substitutional interpretation, all predicate constants are substituends of the bound predicate variables.

Consider, for example, a language L containing '$\in$' as a primitive two-place predicate constant, and suppose we formulate a theory of membership in L with the following as a second-order axiom:

$$(\forall F)(\exists y)(\forall x)[x \in y \leftrightarrow F(x)]. \qquad \textbf{(C)}$$

Now, in nominalism, where predicate quantifiers are interpreted substitutionally, this axiom seems quite plausible as a thesis, stipulating in effect that every predicate expression has an extension. But as plausible a thesis as that might be, it leads directly to Russell's paradox. For, by the nominalist comprehension principle, **(CP!)**,

$$(\exists F)(\forall x)[F(x) \leftrightarrow x \notin x], \qquad \textbf{(D)}$$

is provable under such an interpretation; and, because no predicate quantifier occurs in $x \notin x$, then, by **(UI!$_2$)**, $x \notin x$ represents a predicate expression that can be properly substituted for F in a universal instantiation of **(C)**.

In constructive conceptualism, on the other hand, **(D)** is not an instance of the conceptualist principle, **(CCP!)**, and all that follows by Russell's argument from **(C)** is the fact that '$\in$' cannot stand for a "predicative" (relational) concept. That is, instead of the contradiction that results when predicate quantifiers are interpreted substitutionally, **(C)**, when taken as an axiom of a theory of membership in constructive conceptualism, leads only to the result that the membership predicate does not stand for a "predicative" (relational) concept:

$$\neg(\exists R)(\forall x)(\forall y)[R(x, y) \leftrightarrow x \in y].$$

In other words, as a plausible thesis to the effect that every "predicative" concept has an extension, **(C)** is consistent, not inconsistent, in constructive conceptualism.

On the nominalist strategy, the notion of a "predicative" context is purely grammatical in terms of logical syntax; that is, an open formula is "predicative" in nominalism just in case it contains no bound predicate variables. In constructive conceptualism, the notion of a "predicative" context is semantical, which means that in addition to being "predicative" in nominalism's purely grammatical sense, it must also stand for a "predicative" concept. It is for this reason that the second-order logic of constructive conceptualism must be free of existential presuppositions regarding predicate constants, which is why the binary predicate, '$\in$', in a theory of membership having **(C)** as an axiom, cannot stand for a value of the bound predicate variables.

In general, how we determine which, if any, of the primitive predicate constants of an applied language and theory stand for a "predicative" concept depends on the domain of discourse of that language and theory and how that domain is to be conceptually represented. In particular, those primitive predicates that are to be taken as standing for a "predicative" concept will be stipulated as doing so in terms of the "meaning postulates" of that theory, whereas those that are not will usually occur in axioms that determine that fact.

Identity and its role in a logical theory marks another important difference between nominalism and constructive conceptualism. In nominalism, identity is definable in any applied language with finitely many predicate constants. This is because such a definition can be given in terms of a formula representing indiscernibility with respect to those predicate constants. Suppose, for example L is a language with two just two predicate constants, a one-place predicate constant P, and a two-place predicate constant R. Then, identity can be defined in theories formulated in terms of L as follows:

$$a = b \leftrightarrow [P(a) \leftrightarrow P(b)] \wedge [R(a,a) \leftrightarrow R(b,a)] \wedge [R(a,b) \leftrightarrow R(b,b)] \wedge$$
$$[R(a,a) \leftrightarrow R(a,b)] \wedge [R(b,a) \leftrightarrow R(b,b)]$$

In other words, in any given application based on finitely many predicate constants, which we may assume to be the standard situation, identity, in nominalism, is reducible to a first-order formula, which is why the identity sign is allowed to occur in instances of **(CP!)** under its nominalistic, substitutional interpretation. Such a definition will not suffice in constructive conceptualism, on the other hand, because the first-order formula in question, even were it to stand for a "predicative" concept, cannot justify the substitutivity of identicals in impredicative contexts. The identity sign is not eliminable, or otherwise reducible, in constructive conceptualism, in other words, because, on the basis of Leibniz's law, identity must allow for full substitutivity even in impredicative contexts. Thus, whereas,

$$x = y \leftrightarrow (\forall F)[F(x) \leftrightarrow F(y)],$$

is provable in nominalism's second-order logic, as based on its substitutional interpretation, the right-to-left direction of this same formula is not provable in the logic of constructive conceptualism.

Finally, note that although "impredicative" definitions are not allowed in nominalism, they are not precluded in the logic of constructive conceptualism. The difference is determined by the role free predicate variables have in each of these frameworks. In nominalism, free predicate variables must be construed as dummy schema letters, which in an applied language stand for arbitrary first-order formulas of that theory. This means that the substitution rule,

$$if\ \vdash \psi,\ then\ \vdash \psi[\varphi/G(x_1, ..., x_n)],$$

is valid on the substitutional interpretation only when φ is "predicative" in nominalism's purely grammatical sense, i.e., only when no predicate variable has a bound occurrence in φ. Indeed, the rule must be restricted in this way because, otherwise, by taking ψ to be the following instance of **(CP!)**,

$$(\exists F)(\forall x_1)...(\forall x_n)[F(x_1, ..., x_n) \leftrightarrow G(x_1, ..., x_n)],$$

we would be able to derive the full, unrestricted impredicative comprehension principle, **(CP)**, by substituting φ for G, and, thereby, transcend the substitutional interpretation of predicate quantifiers.

In the predicate logic of constructive conceptualism, on the other hand, the above substitution rule is valid for all formulas, regardless whether or not they contain any bound predicate variables.[13] But, unlike the situation in nominalism, the validity of such a rule does not lead to the unrestricted impredicative comprehension principle. In particular, the above instance of **(CP!)** is not also an instance of **(CCP!)**. This means that the notion of a possible (explicit) definition of a predicate constant is broader in constructive conceptualism than it is in nominalism, where only first-order formulas are allowed as definiens.

Nevertheless, on the basis of the rule of substitution for all formulas, it can be shown that definitions in constructive conceptualism whose definiens contain a bound predicate variable will still be noncreative and will still allow for the eliminability of defined predicate constants.[14] Thus, even though constructive conceptualism validates only a "predicative" comprehension principle, i.e., a comprehension principle encompassing laws of compositionality that are in accordance with the vicious circle principle, it nevertheless allows for impredicative definitions of predicate constants, that is, of predicate constants that do not stand for values of the bound predicate variables and that cannot therefore be existentially generalized upon.

5 Holistic Conceptualism

The difference between nominalism and constructive conceptualism, we have said, is similar to that between standard first-order logic and free logic, i.e., a first-order logic free of existential presuppositions regarding singular terms. The freedom from such presuppositions for predicates in constructive conceptualism indicates how concept-formation is essentially an open process, and that part of that process is a certain tension, or disequilibrium, between the predicates and formulas that stand for concepts at a given stage of concept-formation and those that do not. This disequilibrium in concept-formation is the real driving force of what is known as "ramified" second-order logic, though, strictly speaking, ramified second-order logic is usually associated with nominalism and not with conceptualism.[15]

We can close the "gap" in between predicates that stand for "predicative" concepts at a given stage of concept-formation and those that do not by introducing new predicate quantifiers in addition to the original ones. A new

[13] A similar substitution rule for singular terms is also valid in free logic incidentally, i.e., singular terms can be validly substituted for free object variables even though they cannot be validly substituted for bound object variables.

[14] See Cocchiarella 1986a, chap. 2, sec. 3, for a proof of this claim.

[15] This is because standard "predicative" logic has been associated with nominalism, and standard ramified logic is an extension of standard "predicative" logic, and not of the free "predicative" logic of constructive conceptualism.

comprehension principle would then be added that allowed formulas containing predicate variables bound by the original predicate quantifiers, but not also formulas containing predicate variables bound by the new predicate quantifiers. This will close the "gap" between those formulas that stand for "predicative" concepts at the initial stage and those that do not, because the latter now stand for "predicative" concepts at the new, second stage. But in doing so we open up a new "gap" between the formulas that stand for "predicative" concepts at the new stage and those that do not; and of course we can then go on to close this new "gap" by introducing predicate quantifiers that are new to this stage, along with a similar comprehension principle. This process is what is known as "ramification".

Formally, the process can be described in terms of a potentially infinite sequence of predicate quantifiers $\forall^1, \exists^1, ..., \forall^j, \exists^j, ...$ (for each positive integer j), all of which can be affixed to the same predicate variables. The quantifiers $(\forall^j F)$ and $(\exists^j F)$, where F is an n-place predicate variable, will then be understood to refer to all, or some, respectively, of the n-ary "'predicative"' concepts that can be formed at the *jth* stage of the potentially infinite sequence of stages of concept-formation in question. But because open formulas representing "predicative" contexts of later stages will not be substituends of predicate variables bound by quantifiers of an earlier stage, this means that the logic of the quantifiers $\forall^j$ and $\exists^j$ must be free of existential presuppositions regarding predicate expressions, which is why the comprehension principle for this logic must be closed with respect to all the predicate variables occurring free in the comprehending formula. Thus, as applied at the *jth* stage, the ramified conceptualist comprehension principle that is validated in this framework is the following:

$$(\forall^j G_1)...(\forall^j G_k)(\exists^j F)(\forall x_1)...(\forall x_n)[F(x_1, ..., x_n) \leftrightarrow \varphi], \qquad \textbf{(RCCP!)}$$

where
(1) $G_1, ..., G_k$ are all of the predicate variables occurring free in φ;
(2) F is an n-place predicate variable not occurring free in φ;
(3) $x_1, ..., x_n$ are distinct individual variables; and
(4) φ is a pure ramified formula, i.e., one in which no nonlogical constants occur and in which
(a) the identity sign also does not occur and
(b) in which no predicate variable is bound by a quantifier of a stage $> j$, i.e., for all $i \geq j$, neither $\forall^j$ nor $\exists^j$ occurs in φ.

The process of concept-formation that we are describing here amounts to a type of *reflective abstraction* that involves a projection of previously constructed concepts onto a new plane of thought where they are reorganized under the closure conditions of new laws of concept-formation characteristic of the new stage in question. This pattern of reflective abstraction is precisely what is represented by the ramified comprehension principle **(RCCP!)** and the logic of constructive conceptualism. Each successive stage of concept-formation in the ramified hierarchy is generated by a disequilibrium, or conceptual tension, between the formulas that stand for the "predicative" concepts of the preceding stage, as opposed to those that do not. Thus, in order to close the "gap" between formulas that stand for "predicative" concepts and those do not, we must proceed through a potentially infinite sequence of stages of concept-formation.

Whatever the motivation for ramification in nominalism, it is clear that what moves us on from one stage of concept-formation to the next in constructive conceptualism is a drive for closure, where all predicates stand for concepts. Such a closure cannot be realized in constructive conceptualism, of course, where the principal constraint guiding the formation of "predicative" concepts is their being specifiable by conditions that are in accord with the so-called vicious circle principle. But the particular pattern of reflective abstraction that corresponds to this constraint is not all there is to concept-formation, and, in fact, as a pattern that represents a drive for closure, it contains the seeds of its own transcendence to a new plane of thought where such closure is achieved.

Concept-formation is not constrained by the vicious circle principle, in other words, because after reaching what Piaget calls the stage of formal operational thought, certain new patterns of concept-formation are realizable, albeit usually only in post-adolescence.[16] One such pattern involves an idealized transition to a limit, where "impredicative" concept-formation becomes possible, i.e., where the restrictions imposed by the vicious circle principle are transcended.

> The idealized transition to a limit, in the case of our ramified logic, is conceptually similar to, but ontologically different from, an actual transition to a limit at an infinite stage of concept-formation. This is a stage of concept-formation that, in effect, is not only the summation of all of the finite stages of the ramified hierarchy but also one that is closed with respect to itself.

Ontologically, of course, there cannot be an infinite stage of concept-formation, but that is not to say that an idealized transition to a limit is conceptually impossible as well. Indeed, such an idealized transition to a limit is precisely what is assumed to be possible in *holistic conceptualism*, and it is possible, moreover, on the basis of the pattern of reflective abstraction represented by **(RCCP!*)**. That is, in holistic conceptualism, the drive for closure upon which the pattern of reflective abstraction of ramified constructive conceptualism is based is finally achieved, albeit only as the result of an idealized transition to a limit and not on the basis of an actual transition. In this way, conceptualism, by means of a mechanism of autoregulation that enables us to construct stronger and more complex logical systems out of weaker ones, is able to validate not only the ramified conceptualist comprehension principle but also the full, unqualified "impredicative" comprehension principle **(CP)** of "standard" second-order logic. There is no comparable mechanism in nominalism that can similarly lead to a validation of the impredicative comprehension principle **(CP)**.

> What is inadequate about the logic of constructive conceptualism, it is important to note, is that it cannot provide

[16] Cf. Piaget 1977.

> an account of the kind of impredicative concept-formation that is necessary for the development and use of the theory of real numbers, and which, as a matter of cultural history, we have in fact achieved since the nineteenth century.

The concept of a least upper bound, for example, or of the limit of a converging sequence of rational numbers, is an impredicative concept that was not acquired by the mathematical community until a little more than a century ago; and although, in our own time, it is not usually a part of a person's conceptual repertoire until post-adolescence, nevertheless, with proper training and conceptual development, it is a concept that most of us can come to acquire as a cognitive capacity. Yet, notwithstanding these facts of cultural history and conceptual development, it is also a concept that cannot be accounted for from within the framework of constructive conceptualism.

> The constraints of the vicious circle principle, at least in the way they apply to concept-formation, simply do not conform to the facts of conceptual development in an age of advanced scientific knowledge.

The validation of the full comprehension principle in holistic conceptualism, which we will hereafter refer to simply as conceptualism, does not mean that the logic of constructive conceptualism is no longer a useful part of conceptualism.

> What it does mean is that although all predicates stand for "predicable" concepts, not all predicates stand for "predicative" concepts, and that is a distinction we can turn to in conceptualism whenever it is relevant and useful.

6 The Logic of Nominalized Predicates

Is there no difference then between logical realism and holistic conceptualism as theories of predication, other than the fact that the latter presupposes a logic of "predicative" concepts as a proper part? Well, in fact there is a difference once we consider the import of nominalized predicates and propositional forms as abstract singular terms in the wider context of modal predicate logic. The use of nominalized predicates as abstract singular terms is not only a part of our commonsense framework, but it is also central to how both logical realism and conceptual intensional realism provide an ontological foundation for the natural numbers and other parts of mathematics. This part of logical realism is sometimes called ontological logicism.

In Bertrand Russell's form of logical realism, or ontological logicism, for example, universals are not just what predicates stand for, but also what nominalized predicates, i.e., *abstract nouns*, denote as singular terms.[17] Here, by

[17] Cf. Russell 1903, p. 43.

nominalization we mean the transformation of a predicate phrase into an abstract noun, which is represented in logical syntax as a singular term, i.e., the type of expression that can be substituted for first-order object variables. The following are some examples of predicate nominalizations:

is triangular	⇛	triangularity
is wise	⇛	wisdom
is just	⇛	justice

It was Plato who first recognized the ontological significance of such a transformation and who built his ontology and his account of predication around it. In nominalism, of course, abstract nouns denote nothing.

In English we usually mark the transformation of a predicate into an abstract noun by adding such suffixes as '-ity', '-ness', or 'hood', as with 'triangularity', 'redness', and 'brotherhood'. We do not need to introduce a special operator for this purpose in logical syntax, however. Rather, we need only delete the parentheses that are a part of a predicate variable or constant in its predicative role. Thus, for a monadic predicate F we would have not only formulas such as $F(x)$, where F occurs in its predicative role, but also formulas such as $G(F)$, $R(x, F)$, where F occurs nominalized as an abstract singular term.

> Note: In $F(F)$ and $\neg F(F)$, F occurs both in it predicative role and as an abstract singular term, though in no single occurrence can it occur both as a predicate and as a singular term.

With nominalized predicates as abstract singular terms, it is convenient to have complex predicates represented directly by using Alonzo Church's variable-binding λ-operator. Thus, where φ is a formula of whatever complexity and n is a natural number, we have a complex predicate of the form $[\lambda x_1...x_n\varphi](\ )$, which has parentheses accompanying it in its predicative role, but which are deleted when the complex predicate is nominalized. With λ-abstracts, the comprehension principle can be stated in a stronger and more natural form as

$$(\exists F)([\lambda x_1...x_n\varphi] = F). \qquad (\mathbf{CP}^*_\lambda)$$

This form is stronger than **(CP)** in that it implies, but is not implied by, the latter.[18] One of the rules for the new λ-operator is the rule of λ-conversion,

$$[\lambda x_1...x_n\varphi](a_1, ..., a_n) \leftrightarrow \varphi[a_1/x_1, ..., a_n/x_n] \qquad (\lambda\textbf{-Conv}^*)$$

The grammar of our logical syntax is now more complicated of course. In particular, singular terms and formulas must now be defined simultaneously. For this purpose we will speak of a meaning expression of a given type n, where n is a natural number. We will use 0 to represent the type of *singular terms*

[18] The λ in (CP^*_λ) indicates that a λ-abstract is part of this principle, and the '*' indicates that nominalized predicates may occur in φ as singular terms.

(or just 'terms' for short), 1 to represent the type of *formulas* (propositional forms), and $n+1$ to represent the type of *n-place predicate expressions*.

For each natural number n, we recursively define the meaningful expressions of type n, in symbols, $\mathbf{ME}_n$, as follows:

1. every individual variable (or constant) is in ME_0, and every n-place predicate variable (or constant) is in both $\mathbf{ME}_{n+1}$ and $\mathbf{ME}_0$;
2. if $a, b \in \mathbf{ME}_0$, then $(a = b) \in \mathbf{ME}_1$;
3. if $\pi \in \mathbf{ME}_{n+1}$ and $a_1, ..., a_n \in \mathbf{ME}_0$, then $\pi(a_1, ..., a_n) \in \mathbf{ME}_1$;[19]
4. if $\varphi \in \mathbf{ME}_1$ and $x_1, ..., x_n$ are pairwise distinct individual variables, then $[\lambda x_1...x_n\varphi] \in \mathbf{ME}_{n+1}$;
5. if $\varphi \in \mathbf{ME}_1$, then $\neg\varphi \in \mathbf{ME}_1$;
6. if $\varphi, \chi \in \mathbf{ME}_1$, then $(\varphi \to \chi) \in \mathbf{ME}_1$;
7. if $\varphi \in \mathbf{ME}_1$ and a is an individual or a predicate variable, then $(\forall a)\varphi \in \mathbf{ME}_1$;
8. if $\varphi \in \mathbf{ME}_1$, then $[\lambda\varphi] \in \mathbf{ME}_0$; and
9. if $n > 1$, then $\mathbf{ME}_n \subseteq \mathbf{ME}_0$.

By clause (9), every predicate expression without parentheses is a singular term. This includes 0-place predicates but not formulas in general unless they are of the form $[\lambda\varphi]$, which we take as the nominalization of φ, which we read as 'that φ'. For convenience, however, we shall write '$[\varphi]$' for '$[\lambda\varphi]$'.[20]

It is noteworthy that this logical grammar contains what might described as the essential parts of a theory of logical form: namely,

- (1) the basic forms of predication, as in $F(x)$, $R(x,y)$, etc.,
- (2) propositional (sentential) connectives, e.g., $\wedge$, $\vee$, $\to$, and $\leftrightarrow$,
- (3) quantifiers that reach into predicate as well as subject (or argument) positions,
- (4) nominalized predicates and propositional forms as abstract singular terms.

[19] If $n = 0$, we take $a_1, ..., a_n$ (and similarly $x_1, ..., x_n$) to be the empty sequence, resulting in this case in a 0-place predicate expression, which, as already noted, we take to be a formula.

[20] We could require that n be greater than 1 in clause (4)—in which case $[\lambda\varphi] \in \mathrm{ME}_1$ would not follow—and then have clause (8) state that $[\varphi] \in \mathrm{ME}_0$ when $\varphi \in \mathrm{ME}_1$. But then general principles—such as (CP^*_λ) and ($\Box\mathrm{Ext}^*$) described below—that we want to apply to all n-place predicate expressions would have to be stated separately for $n = 0$.

> These four components correspond to fundamental features of natural language, and each needs to be accounted for in any theory of logical form underlying natural language.

Now one of our goals here is to characterize a consistent second-order predicate logic with nominalized predicates and propositional forms as abstract singular terms. This goal is important because such a logic deals with the four important features of natural language described above. Another goal is that as a framework for logical realism or (holistic) conceptualism such a logic should contain all of the theorems of "standard" second-order predicate logic as a proper part.[21] This means in particular that we should retain all of the theorems of classical propositional logic, and that all instances of the comprehension principle **(CP)** of "standard" second-order logic—i.e., instances in which abstract singular terms do not occur—should be provable. Initially, we will assume standard first-order predicate logic with identity as well; but, as we will see, it may be appropriate to assume "free" first-order predicate logic instead. With standard first-order predicate logic, we have by axiom **(A8)**,

$$(\exists y)(F = y)$$

as provable for every nominalized predicate F, and therefore also for λ-abstracts as well:

$$(\exists y)([\lambda x_1...x_n\varphi] = y).$$

Another consequence is that the first-order principle of universal instantiation now also applies to nominalized predicates as abstract singular terms[22]:

$$(\forall x)\varphi \rightarrow \varphi[F/x] \qquad (\mathrm{UI}_1^*)$$

It would be ideal, of course, if the comprehension principle $(\mathbf{CP}_\lambda^*)$ can be assumed for all formulas φ, including those in which nominalized predicates and propositional forms occur as abstract singular terms. But, if the logic is not "free of existential presuppositions" for singular terms, such an unrestricted second-order logic—which is similar to the system of Gottlob Frege's *Grundgesetze*[23] —is subject to Russell's paradox of predication, and therefore cannot be assumed as a consistent principle. Thus, e.g., where φ represents the Russell property of being identical to a property that is not predicable of itself[24], which as a λ-abstract can be formalized as

$$[\lambda x(\exists G)(x = G \wedge \neg G(x))],$$

[21] By the valid formulas of "standard" second-order logic we mean all of the second-order formulas that are valid with respect to Henkin general models.

[22] We use '(UI_1^*)' to label this principle, with the subscript indicated that it is first-order object quantifier thesis, and with the '*' to indicate that the principle applies to abstract singular terms as well object variables and constants.

[23] Frege's expressions for value-ranges (*Wertverläufe*) were his formal counterparts of predicate nominalizations, i.e., formal counterparts of expressions such as 'the concept F'.

[24] See Russell 1903, p. 97.

then, by the unrestricted comprehension principle $(\mathbf{CP}^*_\lambda)$,

$$(\exists F)([\lambda x(\exists G)(x = G \wedge \neg G(x))] = F) \tag{1}$$

is provable, and therefore, by Leibniz's Law, $(\mathbf{LL}^*)$, so is the weaker form,

$$(\exists F)(\forall x)[F(x) \leftrightarrow [\lambda x(\exists G)(x = G \wedge \neg G(x))](x)], \tag{2}$$

which, by λ-conversion, is equivalent to

$$(\exists F)(\forall x)(F(x) \leftrightarrow (\exists G)[x = G \wedge \neg G(x)]). \tag{3}$$

But by $(\mathbf{UI}^*_1)$,

$$(\forall x)(F(x) \leftrightarrow (\exists G)[x = G \wedge \neg G(x)]) \to (F(F) \leftrightarrow (\exists G)[F = G \wedge \neg G(F)]),$$

and, by $(\mathbf{LL}^*)$,

$$(\exists G)[F = G \wedge \neg G(F)] \leftrightarrow \neg F(F)$$

are also provable, and therefore,

$$(\forall x)(F(x) \leftrightarrow (\exists G)[x = G \wedge \neg G(x)]) \to [F(F) \leftrightarrow \neg F(F)]$$

is provable as well. But, by sentential logic, the consequent of this last conditional is clearly impossible, which means that

$$\neg(\forall x)(F(x) \leftrightarrow (\exists G)[x = G \wedge \neg G(x)]) \tag{4}$$

is provable, and therefore, by $(\mathbf{UG}_2)$ and a quantifier negation law,

$$\neg(\exists F)(\forall x)(F(x) \leftrightarrow (\exists G)[x = G \wedge \neg G(x)]), \tag{5}$$

which contradicts (3) above.

The above result is what is known as Russell's paradox of predication. Russell himself later turned to his theory of ramified types to avoid the contradiction. Later, it was later pointed out that a theory of simple types sufficed, at least for the so-called logical paradoxes such as Russell's. The idea of a hierarchy of types based on the fundamental asymmetry of subject and predicate is fundamentally correct, we maintain. But, as we will see, the idea can be simplified even further within a strictly second-order predicate logic with nominalized predicates as abstract singular terms that is both consistent and equivalent to the simple theory of types.

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